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A Class of Three-Dimensional Compressible Fluid Flows

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I. INTRODUCTION

Even when we restrict ourselves to steady nonviscous nonheat-conducting fluid flow subject to no external (body) forces there is no known general method of solving the differential equations governing this fluid motion without additional assumptions or approximations.

Until about twenty-five years ago there was relatively little motivation for worrying about compressibility effects and thus the additional simplifying assumption that the fluid was incompressible was usually made. Under this assumption, in particular, in the irrotational case where the problem reduces to that of classical potential theory, explicit classes of solutions have been found [1].

On the other hand, as long as fifty years ago an explicit class of solutions, now called simple waves, or Prandtl-Meyer flow, admitting compressibility was obtained for the special case of plane (and also supersonic) flows [2]. In other attempts to include compressibility linearization or perturbation approximations have been used [3]. Other approaches have employed the "hodograph method" (see, for example, ref. 4). Finally, several "degenerate" three dimensional flows (see, for example, ref. 5), and a generalized Prandtl-Meyer flow [6] have been found.

Rather than confine ourselves to plane flow or incompressible flow, or apply any of the other common assumptions or approximations, we require that the divergence of the unit tangent vector of the streamlines vanish. This purely geometrical condition is suggested by making use of an intrinsic formulation of the fluid flow equations. Isolated examples of intrinsic descriptions are not new [7], but only fairly recently has an extensive and systematic exploitation of intrinsic methods been carried out by Coburn [8]. It will be shown that this condition on the unit tangent vector of the stream lines has some significance for the relation between incompressible and compressible flows.

A class of solutions in explicit form (helical flows) which satisfy the condition that the divergence of the unit tangent vector of the streamlines vanish has already been found [9]. Invoking this condition we now obtain another class of three-dimensional compressible flows. Again, because the original system of differential equations reduces to a pair of first order ordinary differential equations, the solutions appear in explicit form.

The main development consists of starting in section 2 with a standard form of the equations of fluid flow and deriving in succession several sets of necessary conditions. It is shown at each step in this sequence that the derived conditions are also sufficient so that functions satisfying the final set of conditions in Section VI do indeed yield fluid flows. The flows are such that the streamlines lie on surfaces of revolution.

II. The Basic Equations

The fluid flows to be considered here will be restricted to those which satisfy the following differential equations in which v^i is the velocity, ρ is the density, p is the pressure, S is the entropy, $\mathfrak{P}(p)$ and $\mathfrak{S}(S)$ are given functions of p and S respectively, and ∇_i denotes covariant differentiation:

$$\rho v^i \nabla_i v_j = - \nabla_j p \qquad \text{(eq. of motion)} \tag{1}$$

$$\nabla_i \rho v^i = 0 \qquad \text{(continuity eq.)} \tag{2}$$

$$\rho = \mathfrak{P}(p) \cdot \mathfrak{S}(S) \qquad \text{(eq. of state)} \tag{3}$$

$$v^i \nabla_i S = 0. \tag{4}$$

The last equation says that the entropy is constant along streamlines. Thus, these are nonviscous, nonheat-conducting steady flows which are subject to no external forces [10]. Note that the equation of state admits polytropic gases [11].

If t^i designates the unit tangent vector along the streamlines, and q the magnitude of the velocity, then $v^i = qt^i$ and with this substitution Eqs. (1) and (2) become respectively

$$(\rho q t^i \nabla_i q)\, t_j + \rho q^2 \kappa n_j = - \nabla_j p \tag{5}$$

$$t^i \nabla_i \ln \rho q + \nabla_i t^i = 0 \tag{6}$$

where κ and n_j are the curvature and unit principal normal vector respectively of the streamline congruence. Forming the scalar products of Eq. (5) with t^j,

n^j, b^j where b^j is the unit binormal vector of the stream lines, Eq. (5) takes the form, by use of

$$P(p) = \int_{p_0}^{p} \frac{1}{\mathfrak{P}(\bar{p})}\, d\bar{p}$$

and Eqs. (3) and (4),

$$t^i \nabla_i P = - t^i \nabla_i \frac{q^2 \mathfrak{S}}{2} \tag{7}$$

$$n^i \nabla_i P = - q^2 \mathfrak{S} \kappa \tag{8}$$

$$b^i \nabla_i P = 0. \tag{9}$$

Using Eqs. (3) and (4) we can write Eq. (6) in the form

$$2t^i \nabla_i \ln \mathfrak{P} + t^i \nabla_i \ln q^2 \mathfrak{S} + 2\nabla_i t^i = 0.$$

But

$$2t^i \nabla_i \ln \mathfrak{P} = 2\frac{d \ln \mathfrak{P}}{dP}\, t^i \nabla_i P = - \frac{d\mathfrak{P}}{dp}\, t^i \nabla_i\, q^2 \mathfrak{S} = - \frac{d\mathfrak{P}}{dp}\, q^2 \mathfrak{S}\, t^i \nabla_i \ln q^2 \mathfrak{S}$$

so that Eq. (6) takes the form

$$\left(1 - q^2 \mathfrak{S} \frac{d\mathfrak{P}}{dp}\right) t^i \nabla_i \ln q^2 \mathfrak{S} + 2\nabla_i t^i = 0. \tag{10}$$

Note that the four scalar equations (7)-(10) are equivalent to the original system, Eqs. (1)-(4), since with a solution consisting of a unit vector t^i and functions P and $q^2\mathfrak{S}$ one can go backwards and obtain a solution of the original system. Specifically, if t^i, P, $q^2\mathfrak{S}$ satisfy Eqs. (7)-(10), if $\mathfrak{S}$ is any function that satisfies $t^i \nabla_i \mathfrak{S} = 0$, and if ρ is defined by Eq. (3) in which p is the inverse of the function defined by the integral above, then in addition to Eqs. (3) and (4), with $v^i = qt^i$, Eqs. (1) and (2) will also be satisfied.

III. The Condition $\nabla_i t^i = 0$

From Eq. (10) the condition

$$\nabla_i t^i = 0 \tag{11}$$

implies that either

$$t^i \nabla_i \ln q^2 \mathfrak{S} = 0 \tag{12}$$

or

$$1 - q^2 \mathfrak{S} \frac{d\mathfrak{P}}{dp} = 0. \tag{13}$$

From Eqs. (7) and (4) respectively we see that Eq. (12) implies that p and q are constant along streamlines. On the other hand, since $\mathfrak{S}(d\mathfrak{P}/dp)$ is the reciprocal of the square of the velocity of sound, Eq. (13) corresponds to flows in which the Mach number is one everywhere. While such flows are of some interest,[1] they will be excluded in the following considerations, and the condition $\nabla_i t^i = 0$ will be identified with Eq. (12).

The condition $\nabla_i t^i = 0$ is a purely geometrical condition on the congruence of streamlines which says, roughly, that the streamlines do not diverge (or converge). In particular, it can be shown that if there exists a family of parallel stream surfaces on each of which the streamlines form a family of parallel curves then $\nabla_i t^i = 0$. For example, helices on concentric circular cylinders form a congruence for which $\nabla_i t^i = 0$ and fluid flows whose streamlines have this geometry are described in ref. 9.

Now let us consider for a moment the case of *incompressible* fluid flows. Equations (1) and (2) are valid with ρ = constant. Proceeding from these equations as before one obtains a system analogous to Eqs. (7)-(10), namely,

$$t^i \nabla_i \frac{p}{\rho} = - t^i \nabla_i \frac{q^2}{2} \tag{14}$$

$$n^i \nabla_i \frac{p}{\rho} = - q^2 \kappa \tag{15}$$

$$b^i \nabla_i \frac{p}{\rho} = 0 \tag{16}$$

$$t^i \nabla_i \ln q^2 + 2\nabla_i t^i = 0. \tag{17}$$

Suppose we have a solution of the system of equations (14)-(17) in which $\nabla_i t^i = 0$. If we use the same t^i (and hence the same n^i, b^i, κ) in Eqs. (7)-(10) for *compressible* flows and if we substitute for P and $q^2\mathfrak{S}$ in these equations the known functions p/ρ and q^2, respectively, then these equations will be satisfied. Moreover, if p_{I} designates the pressure of the incompressible flow, and p_{C} the pressure of the compressible flow, then since

$$\frac{dP}{dp_{\mathrm{C}}} \nabla_i p_{\mathrm{C}} = \nabla_i P = \nabla_i \frac{p_{\mathrm{I}}}{\rho} = \frac{1}{\rho} \nabla_i p_{\mathrm{I}}$$

it follows that the unit normal vector of the pressure surfaces at each point is the same for both flows and thus the two families of surfaces are the same. On the other hand, if we have a solution of Eqs. (7)-(10) in which $\nabla_i t^i = 0$

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[1] See ref. 3. These flows satisfy a Chaplygin-Karman-Tsien type of equation of state.

we obtain (if we exclude the special case described above) a solution of Eqs. (14)-(17). In brief, we thus have the result:

Corresponding to each incompressible (compressible) flow with $\nabla_i t^i = 0$ *there is a compressible (incompressible) flow having the same streamlines and constant pressure surfaces.*

The converse of this result is also true. That is, *if an incompressible and a compressible flow have the same (nonstraight) streamlines and constant pressure surfaces, then they both have* $\nabla_i t^i = 0$. In order to show this it is convenient to write Eq. (17) in the form

$$t^i \nabla_i \frac{p}{\rho} - q^2 \nabla_i t^i = 0 \tag{18}$$

and Eq. (10) in the form

$$\left(1 - q^2 \mathfrak{S} \frac{d\mathfrak{P}}{dp}\right) t^i \nabla_i P - q^2 \mathfrak{S}\, \nabla_i t^i = 0. \tag{19}$$

Moreover, since these relations are to be compared it will be necessary now to distinguish notationally between the incompressible velocity magnitude, q_I, and the compressible velocity magnitude, q_C. From our hypothesis that the pressure surfaces are the same for the two flows it follows that the unit normal vector of the pressure surfaces at each point is the same for both flows, which implies $\nabla_i p_\mathrm{I} = \lambda \nabla_i p_\mathrm{C}$ and hence

$$\nabla_i P = \mu \nabla_i \frac{p}{\rho} \tag{20}$$

where λ and μ are scalar functions. Eliminating $t^i \nabla_i P$ and $t^i \nabla_i (p/\rho)$ among Eqs. (18)-(20) we get

$$q_\mathrm{C}^2 \mathfrak{S}\, \nabla_i t^i = \mu \left(1 - q_\mathrm{C}^2 \mathfrak{S} \frac{d\mathfrak{P}}{dp}\right) q_\mathrm{I}^2 \Delta_i t^i. \tag{20a}$$

Similarly, from the hypothesis that both flows have the same nonstraight streamlines and the use of Eqs. (8), (15), and (20) we obtain

$$\mu q_\mathrm{I}^2 = q_\mathrm{C}^2 \mathfrak{S}. \tag{21}$$

Our conclusion follows directly from Eqs. (20a) and (21).

IV. A System Equivalent to Eqs. (7)-(10) when $\nabla_i t^i = 0$

One can express Eqs. (8)-(10) in terms of the variable $B = P + (q^2 \mathfrak{S}/2)$ rather than $q^2 \mathfrak{S}$. (Note that B is constant along streamlines by Eq. (7)). If one does this, then from the integrability conditions of Eqs. (8)-(10) for the function P, one finds that if $\nabla_i t^i = 0$ then there exists a family of stream surfaces on which the streamlines are geodesics [12].

We will want to make use of this special family of stream surfaces very shortly. However, we can, for the moment, admit more generality and let $\{\omega = \text{const.}\}$ be any family of stream surfaces (i.e., an integral of $t^i \nabla_i \omega = 0$) with unit normal N^i. Taking components of Eq. (5) in the directions t^i, $'n^i$, N^i where $'n^i$ is orthogonal to both t^i and N^i we get instead of Eqs. (8) and (9)

$$'n^i \nabla_i P = -q^2 \mathfrak{G}\, '\kappa \tag{22}$$

$$N^i \nabla_i P = -q^2 \mathfrak{G} \kappa_N \tag{23}$$

where κ_N and $'\kappa$ are respectively the normal and geodesic curvatures of the streamlines on $\{\omega = \text{const.}\}$.

Now let x^1, x^2, x^3 be a coordinate system containing the family of stream surfaces $\{\omega = \text{const.}\}$. With $x^3 = \omega$ we get

$$N_i = \left(0, 0, \frac{1}{\sqrt{g^{33}}}\right)$$

and

$$N^i = \frac{g^{i3}}{\sqrt{g^{33}}}$$

where g_{ij} and g^{ij} are covariant and contravariant components of the metric tensor. Further, with the streamlines on $\{\omega = \text{const.}\}$ as the coordinate curves x^1 variable we get

$$t^i = \left(\frac{1}{\sqrt{g_{11}}}, 0, 0\right)$$

$$t_i = \frac{g_{i1}}{\sqrt{g_{11}}}$$

and

$$'n^i = \frac{\epsilon^{ijk}}{\sqrt{g}} N_j t_k = \frac{1}{\sqrt{g_{11} g^{33} g}} (-g_{12}, g_{11}, 0)$$

where $g = |g_{ij}|$. Substituting these expressions for the unit vectors in terms of the metric coefficients into the system of equations (7), (22), (23), (11), (12) and solving the second and third of these for $\partial P/\partial x^2$, $\partial P/\partial x^3$ we get

$$\frac{\partial}{\partial x^1}\left(P + \frac{q^2 \mathfrak{S}}{2}\right) = 0 \tag{24}$$

$$\frac{\partial}{\partial x^2} P = -q^2 \mathfrak{S} \left(\frac{g^{33} g}{g_{11}}\right)^{1/2} {}'\kappa \tag{25}$$

$$\frac{\partial}{\partial x^3} P = - q^2 \mathfrak{S} \frac{- g^{23} \left(\dfrac{g}{g_{11}}\right)^{1/2} {}'\kappa + \kappa_N}{(g^{33})^{1/2}} \tag{26}$$

$$\frac{\partial}{\partial x^1} \left(\frac{g}{g_{11}}\right)^{1/2} = 0 \tag{27}$$

$$\frac{\partial}{\partial x^1} q^2 \mathfrak{S} = 0. \tag{28}$$

In order to simplify Eq. (27), we introduce the metric coefficients ${}'g_{11}$, ${}'g_{12}$, ${}'g_{22}$ of any surface, $\omega = x^3 =$ constant. Evidently,

$$g_{11} = {}'g_{11}, \qquad g_{12} = {}'g_{12}, \qquad g_{22} = {}'g_{22}. \tag{27a}$$

Further, we note that

$$g g^{33} = g_{11} g_{22} - (g_{12})^2. \tag{27b}$$

Again, by definition, we find

$${}'g^{22} = \frac{{}'g_{11}}{{}'g_{11} {}'g_{22} - ({}'g_{12})^2}. \tag{27c}$$

From (27a)-(27c), we see that

$$\frac{g_{11}}{g} = {}'g^{22} g^{33} \tag{27d}$$

so that Eq. (27) can be written

$$\frac{\partial}{\partial x^1} ({}'g^{22} g^{33})^{1/2} = 0. \tag{27e}$$

Finally, we use the family $\{\omega = \text{const.}\}$ mentioned at the beginning of this section on which the streamlines are geodesics (i.e., ${}'\kappa = 0$). Equations (24)-(26), (27e), (28) reduce to

$$\frac{\partial P}{\partial x^1} = 0 \qquad \frac{\partial q^2 \mathfrak{S}}{\partial x^1} = 0 \qquad \frac{\partial P}{\partial x^2} = 0$$

$$\frac{\partial P}{\partial x^3} = - q^2 \mathfrak{S} \frac{\kappa_N}{\sqrt{g^{33}}} \qquad \frac{\partial}{\partial x^1} ({}'g^{22} g^{33}) = 0$$

from which we conclude that P is a function of x^3 alone and $q^2 \mathfrak{S}$, $B = P + (q^2 \mathfrak{S}/2)$, $\kappa_N/\sqrt{g^{33}}$, ${}'g^{22} g^{33}$ are not functions of x^1. Thus we have the result that if $\nabla_i t^i = 0$, then Eqs. (7)-(10) lead to Eqs. (24)-(28) and hence

$$P = P(x^3) \tag{29}$$

$$\frac{\kappa_N}{\sqrt{g^{33}}} = R(x^2, x^3) \tag{30}$$

$$'g^{22}g^{33} = Q^2(x^2, x^3). \tag{31}$$

On the other hand, suppose we start with an arbitrary family of surfaces in Euclidean three-space. Suppose we then construct a coordinate system for this space with these surfaces as $\{x^3 = \text{const.}\}$, and with a family of geodesics on them as the x^1 variable curves. If, further, Eqs. (30) and (31) can be satisfied, then t^i, the unit tangent vector along the geodesics, P, an arbitrary (differentiable) function of x^3, and, $q^2\mathfrak{S} = -(1/R)(dP/dx^3)$ satisfy Eqs. (24)-(28) and hence Eqs. (7)-(10).

V. Equations (30) and (31) for a Family of Surfaces of Revolution

If we are given $f(u, \omega)$, then for each value of ω, the system

$$\begin{aligned} x &= u \cos v \\ y &= u \sin v \\ z &= f(u, \omega) \end{aligned} \tag{32}$$

defines a surface of revolution. On each of these surfaces of revolution a two parameter family of geodesics is given by

$$v = I \int \frac{\sqrt{1 + (f_u(u, \omega))^2}}{u\sqrt{u^2 - I^2}}\, du - J \tag{33}$$

where I and J are arbitrary constants [13]. (Subscripts will denote partial derivatives.) We can pick out a one parameter family by restricting I and J to be functions of a single variable ψ. Doing this for each surface of the family $\{\omega = \text{const.}\}$ corresponds to prescribing I and J as functions of ψ and ω; i.e., we put $I = I(\psi, \omega)$ and $J = J(\psi, \omega)$. Now with these functions in Eq. (33) if we substitute it into Eqs. (32), eliminating v, we get

$$\begin{aligned} x &= u \cos [W(u, \psi, \omega) - J(\psi, \omega)] \\ y &= u \sin [W(u, \psi, \omega) - J(\psi, \omega)] \\ z &= f(u, \omega) \end{aligned} \tag{34}$$

where

$$W(u, \psi, \omega) = I(\psi, \omega) \int \frac{\sqrt{1 + (f_u)^2}}{u\sqrt{u^2 - I^2}} du. \tag{35}$$

Equations (34) describe a transformation of coordinates between x, y, z and u, ψ, ω. In the system u, ψ, ω, the coordinate curves $\{\psi = \text{const.}, \omega = \text{const.}\}$ are geodesics on the surfaces $\{\omega = \text{const.}\}$. Thus the coordinate system u, ψ, ω with $x^1 = u$, $x^2 = \psi$, $x^3 = \omega$ is the type for which we want to satisfy Eqs. (30) and (31).

For this coordinate system we find (see Appendix, Eqs. (A.4)-(A.6))

$$'g^{22} = \frac{1}{(u^2 - I^2)(W_\psi - J_\psi)^2} \tag{36}$$

$$g^{33} = \frac{1 + (f_u)^2}{(f_\omega)^2}. \tag{37}$$

The normal curvature κ_N of the $u =$ variable coordinate curves may be obtained from the Euler formula[2] $\kappa_N = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$ in which the principal normal curvatures κ_1, κ_2 and the angle θ between the coordinate curves are computed from the first and second fundamental forms of the $\omega = \text{const.}$ surfaces.[3] Carrying out this computation one gets (see Appendix, Eqs. (A.7)-(A.10))

$$\kappa_N = \frac{I^2 f_u}{u^3[1 + (f_u)^2]^{1/2}} + \frac{(u^2 - I^2) f_{uu}}{u^2[1 + (f_u)^2]^{3/2}}. \tag{38}$$

Thus we have $'g^{22}$, g^{33}, κ_N in terms of three arbitrary functions f, I^2, J. Substituting Eqs. (36)-(38) into Eqs. (30) and (31), the problem becomes one of solving

$$\frac{I^2 f_u f_\omega}{u^3[1 + (f_u)^2]} + \frac{(u^2 - I^2) f_{uu} f_\omega}{u^2[1 + (f_u)^2]^2} = R(\psi, \omega) \tag{39}$$

$$\frac{1 + (f_u)^2}{(u^2 - I^2)(f_\omega)^2 (W_\psi - J_\psi)^2} = Q^2(\psi, \omega) \tag{40}$$

for, f, I^2, J, R, Q^2.

Functions f, I^2, J from a solution of Eqs. (39) and (40) will specify the coordinate transformation Eq. (34) and, in particular, the two parameter family $\{u = x^1 = \text{variable}\}$ of coordinate curves. These curves may be represented by

$$z = f(u, \omega) \tag{41}$$

[2] See ref. 13, p. 240.

[3] See ref. 13, p. 227.

from Eqs. (32), and

$$v = W(u, \psi, \omega) - J(\psi, \omega) \tag{42}$$

from Eqs. (33) and (35). Note that v is simply the polar angle in the xy plane. Since by construction these curves are geodesics on the surfaces $\{\omega = x^3 = \text{const.}\}$, it follows, as noted at the end of Section IV, that these curves are the streamlines of a fluid flow. The fluid flow will be described completely when the thermodynamic vairables are obtained. As noted at the end of Section IV, we may take P to be an arbitrary function of ω, and take

$$q^2\mathfrak{S} = -\frac{1}{R(\psi, \omega)}\frac{dP}{d\omega} \tag{43}$$

VI. Reduction of Eqs. (39) and (40) to a Pair of First Order Ordinary Differential Equations

First of all, it can be shown by differentiation of (39) with respect to u that I^2, and consequently W and R, cannot be a function of ψ. Equations (39) and (40) then reduce to

$$\frac{I^2 f_u f_\omega}{u^3[1+(f_u)^2]} - \frac{(u^2 - I^2) f_{uu} f_\omega}{u^2[1+(f_u)^2]^2} = R(\omega) \tag{44}$$

$$\frac{1+(f_u)^2}{(u^2 - I^2)(f_\omega)^2} = \bar{Q}^2(\omega) \tag{45}$$

where

$$\bar{Q}^2(\omega) = Q^2(\psi, \omega)\, J_\psi^2(\psi, \omega). \tag{46}$$

Next, after eliminating f_ω between Eqs. (44) and (45) and noting that the resulting equation is linear in $f_u/(1+f_u^2)^{1/2}$ one can perform an integration with respect to u, thereby introducing an arbitrary function η of ω. One then obtains explicit expressions for f_u and f_ω in terms of u, I^2, $\bar{Q}$, η, and $\beta = -R\bar{Q}/2$. Introducing the notation $F = f_u$ and $G = f_\omega$ these are

$$F(u, \omega) = \frac{u(\beta u^2 + \eta)}{\sqrt{u^2 - I^2 - u^2(\beta u^2 + \eta)^2}} \tag{47}$$

$$G(u, \omega) = \frac{-1}{\bar{Q}\sqrt{u^2 - I^2 - u^2(\beta u^2 + \eta)^2}} \cdot \tag{48}$$

The integrability condition of Eqs. (47) and (48) leads to a quadratic equation in u^2. The condition that the coefficients must vanish gives (see Appendix, Eq. (A.11))

$$
\begin{aligned}
-2I^2 \frac{d\eta}{d\omega} + \eta \frac{dI^2}{d\omega} &= \frac{2(1-\eta^2)}{\bar{Q}} \\
2\frac{d\eta}{d\omega} + \beta \frac{dI^2}{d\omega} - 2I^2 \frac{d\beta}{d\omega} &= -\frac{8\eta\beta}{\bar{Q}} \\
\frac{d\beta}{d\omega} &= -\frac{3\beta^2}{\bar{Q}}.
\end{aligned} \tag{49}
$$

Finally, if we solve the last equation for β, substitute this quantity into the first two equations, solve these for $d\eta/d\omega$, $dI^2/d\omega$, and then make a change of independent variable according to $dt/d\omega = -\beta/\bar{Q}$, we obtain

$$X\frac{dX}{dt} = 9X^2 - 4XY - 1 \tag{50}$$

$$X\frac{dY}{dt} = 3XY + 1 \tag{51}$$

where $X = \beta I^2 + \eta$ and $Y = \eta$.

It is important of course that we be able to start, alternatively, with Eqs. (50) and (51) and obtain eqs. (39) and (40). This can be done in the following three formal steps.

(i) Let $X(t)$, $Y(t)$ satisfy Eqs. (50) and (51). Define $\xi(\omega) = X(t(\omega))$, $\eta(\omega) = Y(t(\omega))$, $\beta(\omega) = Ce^{3t(\omega)}$, and $I^2(\omega) = (\xi - \eta)/\beta$ where C is an arbitrary constant and $t(\omega)$ is an arbitrary (differentiable) function of ω. Then η, I^2, β, and $\bar{Q} = -Ce^{3t}/t'$ satisfy Eqs. (49) where $t' = dt/d\omega$.

(ii) Let $\eta, I^2, \beta, \bar{Q}$ satisfy Eqs. (49). Define functions F and G by Eqs. (47) and (48). Then F and G satisfy $F_\omega = G_u$ and $f = \int F\,du + G\,d\omega$, I^2, $\bar{Q}$, and $R = -2\beta/\bar{Q}$ satisfy Eqs. (44) and (45).

(iii) Let $f, I^2, \bar{Q}, R$ satisfy Eqs. (44) and (45). Let Q^2 and J satisfy Eq. (46). Then f, I^2, J, R, Q^2 satisfy Eqs. (39) and (40).

In order to get a fluid flow we may substitute $I^2 = (\xi - \eta)/\beta$ as given in (i) into f as given in (ii) and get

$$f(u,\omega) = \int \frac{u(\beta u^2 + \eta)\,du}{\sqrt{u^2 - \dfrac{\xi-\eta}{\beta} - u^2(\beta u^2+\eta)^2}} - \frac{d\omega}{\bar{Q}\sqrt{u^2 - \dfrac{\xi-\eta}{\beta} - u^2(\beta u^2+\eta)^2}}. \tag{52}$$

With this I^2 and f, the function $W(u, \omega)$ becomes, by substituting (47) into (35),

$$W(u, \omega) = \sqrt{\frac{\xi - \eta}{\beta}} \int \frac{du}{u\sqrt{u^2 - \dfrac{\xi - \eta}{\beta} - u^2(\beta u^2 + \eta)^2}}. \tag{53}$$

Thus f and W can be computed once we have solved the system of equations (50) and (51) and have chosen C and $t(\omega)$. From the definition of R in (ii) and of β and $\bar{Q}$ in (i) one may write Eqs. (43) in the form

$$q^2\mathfrak{S} = -\tfrac{1}{2}\frac{dP}{d\omega}\Big/\frac{dt}{d\omega} \tag{54}$$

so that, effectively, the freedom permitted in the choice of t is reflected in the fact that both P and $q^2\mathfrak{S}$ may be chosen (independently of one another) as arbitrary functions of ω. Note finally, that q and $\mathfrak{S}$ are each required to be constant along streamlines, but both q and $\mathfrak{S}$ could vary from streamline to streamline on $\omega = \text{const.}$

VII. Examples

1. Let $J = \psi$, $t(\omega) = \omega$, $C = 0.01$ and let $X(t)$, $Y(t)$ be the solution of Eqs. (50) and (51) in the neighborhood of $(-0.9, -1)$. With these values and along a curve $\omega = \text{const.}$ the right side of Eq. (52) reduces to

$$5e^{-(3/2)\omega} \int \frac{(w + Y)\,dw}{\sqrt{-w^3 - 2Yw^2 + (1 - Y^2)\,w + Y - X}}$$

where $w = \beta u^2$. To go from one $\omega = \text{const.}$ curve to another we can go along $w + Y = 0$ for which, in this example, the right side of Eq. (52) reduces to

$$10 \int \frac{d\omega}{e^{(3/2)\omega}\sqrt{-X}}.$$

Putting $\omega = -0.05,\ 0,\ 0.05$ in the integrand of the first integral, choosing appropriate limits for both integrals, and performing numerical integrations we get sections $z = f(u, -0.05)$, $z = f(u, 0)$, $z = f(u, 0.05)$ of $\omega = \text{const.}$ surfaces as illustrated in Fig. 1. The right side of Eq. (53) may be written in the form

$$\frac{\sqrt{X - Y}}{2} \int \frac{dw}{w\sqrt{-w^3 - 2Yw^2 + (1 - Y^2)\,w + Y - X}}.$$

Putting $\omega = -0.05, 0, 0.05$ in this we get projections $v = W(u, -0.05) - \psi$, $v = W(u, 0) - \psi$, $v = W(u, 0.05) - \psi$ in the xy plane of the streamlines on the corresponding $\omega =$ const. surfaces as illustrated in Fig. 2.

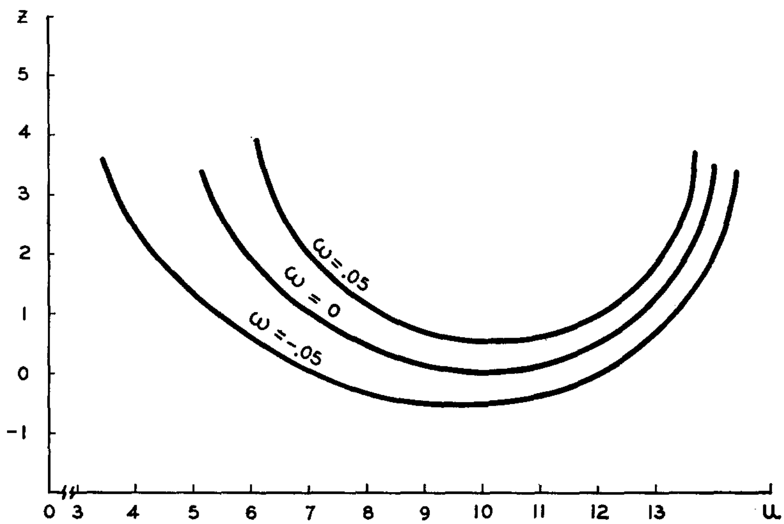

FIG. 1. Sections of stream surfaces of example 1.

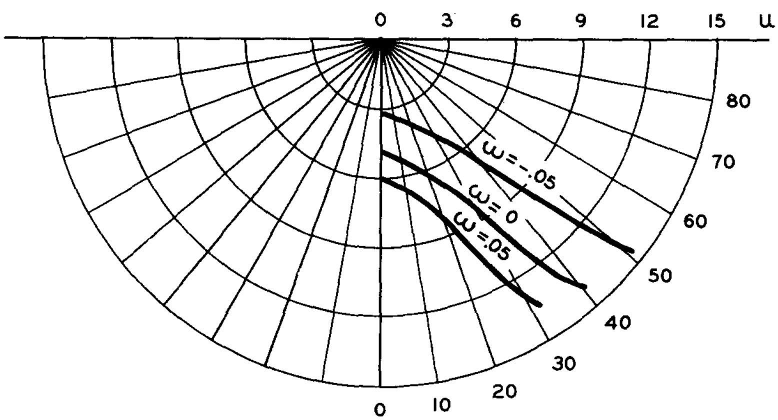

FIG. 2. Projections of stream lines of example 1.

2. Let J, $t(\omega)$, and C be the same as in example (1), but let $X(t)$, $Y(t)$ be the solution of Eqs. (50) and (51) in the neighborhood of (0.5, 0.4). The integral for $f(u, \omega)$ along $\omega =$ const. is the same as in example (1). To go from one $\omega =$ const. curve to another in this example we cannot use the

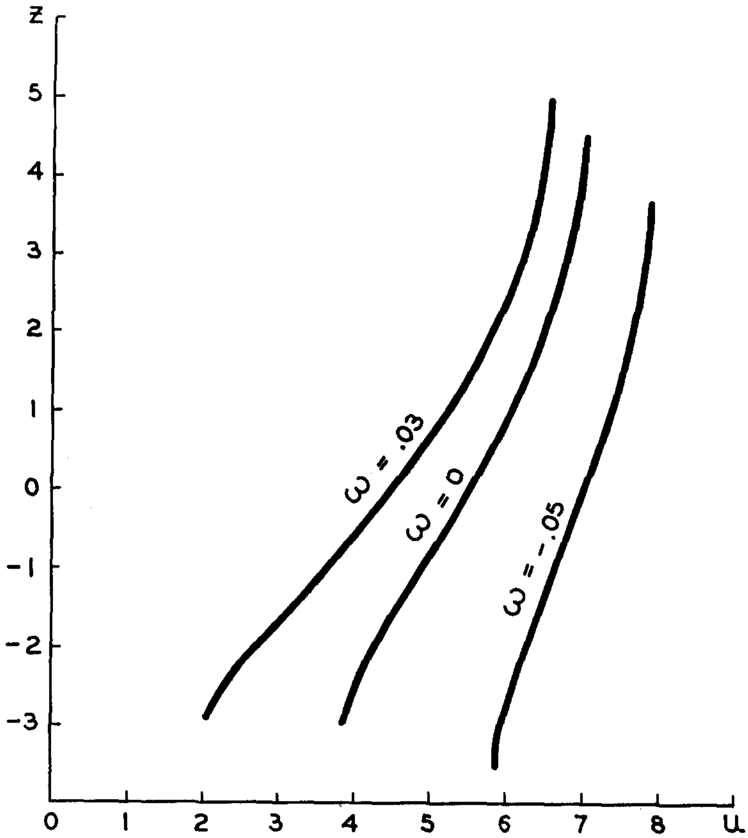

FIG. 3. Sections of stream surfaces of example 2.

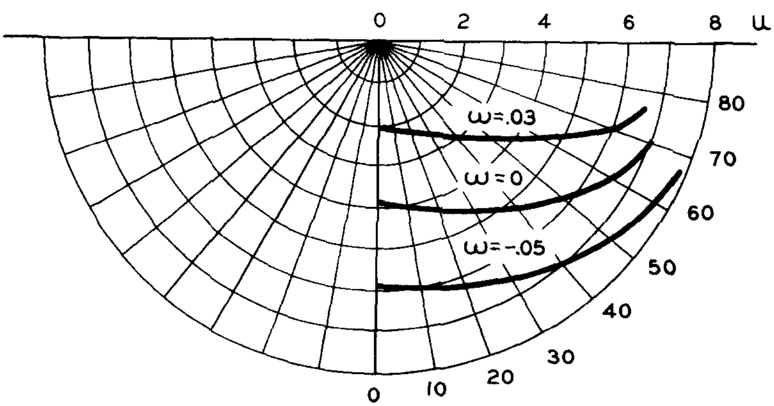

FIG. 4. Projections of stream lines of example 2.

path we used in example 1. In this example we go along $w + Y = \Lambda$, where Λ is a constant, and we get

$$10 \int \frac{1 - \frac{3}{2}\Lambda^2 - (\Lambda/2)\, X^{-1}}{e^{(3/2)\omega}\, \sqrt{\Lambda - \Lambda^3 + \Lambda^2 Y - X}}\, d\omega$$

instead of the second integral in example (1). Sections of $\omega =$ const. surfaces for this example are shown in Fig. 3. Projections in the xy plane of the streamlines on the $\omega =$ const. surfaces are obtained by means of the same integral for $W(u, \omega)$ as in example (1). These are shown in Fig. 4.

Appendix

Here we shall list: (1) the metric coefficients, g_{ij}, and the components $'g^{22}$ and g^{33} associated with the $x^1 = u$, $x^2 = \psi$, $x^3 = \omega$ coordinate system defined by (34); (2) the principal normal curvatures κ_1, and κ_2 of the $\omega =$ const. surfaces; (3) the angle θ between the $u =$ variable coordinate curves and the curves along one of the principal normal directions at any point; (4) the integrability condition for Eqs. (47) and (48). The notation will be that of Eisenhart [13]; the computations are direct but lengthy and the results are listed to aid the reader.

By use of the definition

$$g_{ij} = \frac{\partial x}{\partial x^i}\frac{\partial x}{\partial x^j} + \frac{\partial y}{\partial x^i}\frac{\partial y}{\partial x^j} + \frac{\partial z}{\partial x^i}\frac{\partial z}{\partial x^j} \tag{A.1}$$

and the definition

$$H^2 = \frac{1 + f_u^2}{u^2 - I^2} \tag{A.2}$$

we find, using Eq. (32), that the matrix of g_{ij} is

$$g_{ij} = \begin{pmatrix} u^2H^2 & uIHv_\psi & uIHv_\omega + f_u f_\omega \\ uIHv_\psi & u^2v_\psi^2 & u^2v_\psi v_\omega \\ uIHv_\omega + f_u f_\omega & u^2v_\psi v_\omega & u^2v_\omega^2 + f_\omega^2 \end{pmatrix} \tag{A.3}$$

Substituting from (A.3) into (27c) we find

$$'g^{22} = (u^2 - I^2)^{-1}\,(v_\psi)^{-2}. \tag{A.4}$$

Further, the determinant g of the matrix (A.3) is

$$g = (uf_\omega v_\psi)^2. \tag{A.5}$$

Using (A.3) and (A.5) we find

$$g^{33} = \frac{1 + f_u^2}{f_\omega^2}. \tag{A.6}$$

The principal curvatures of a surface of revolution are given by Eisenhart (cf. [13, p. 227]). In our notation these are

$$\kappa_1 = \frac{f_u}{u[1 + f_u^2]^{1/2}} \tag{A.7}$$

the normal curvature of the parallels $u = $ const., and

$$\kappa_2 = \frac{f_{uu}}{(1 + f_u^2)^{3/2}} \tag{A.8}$$

he normal curvature of the meridional curves $v = $ const.

The angle θ between the coordinate curves on $\omega = $ const. is given by

$$\cos^2\theta = \frac{{}'g_{12}^2}{{}'g_{11}\,{}'g_{22}}. \tag{A.9}$$

Since one family of coordinate curves consists of the parallels and these are lines of curvature (cf. [13, p. 230]) this is the angle we need in Euler's formula for the normal curvature of the $u = $ variable coordinate curves. From (A.3) and the fact that $g_{11} = {}'g_{11}$, $g_{12} = {}'g_{12}$, and $g_{22} = {}'g_{22}$ on $\omega = $ const., we get from (A.9)

$$\cos^2\theta = \frac{I^2}{u^2}. \tag{A.10}$$

To obtain the integrability condition for Eqs. (47) and (48) we calculate

$$\frac{\partial F}{\partial \omega} = \frac{\left\{ \begin{array}{l} [u^2 - I^2 - u^2(\beta u^2 + \eta)^2]\, u(\beta_\omega u^2 + \eta_\omega) \\ \qquad + u(\beta u^2 + \eta)[II_\omega + u^2(\beta u^2 + \eta)(\beta_\omega u^2 + \eta_\omega)] \end{array} \right\}}{[u^2 - I^2 - u^2(\beta u^2 + \eta)^2]^{3/2}}$$

and

$$\frac{\partial G}{\partial u} = \frac{u - u(\beta u^2 + \eta)^2 - 2u^3\beta(\beta u^2 + \eta)}{\bar{Q}[u^2 - I^2 - u^2(\beta u^2 + \eta)^2]^{3/2}}.$$

Equating these and removing common factors we get

$$(u^2 - I^2)(\beta_\omega u^2 + \eta_\omega) + II_\omega(\beta u^2 + \eta) = \frac{1 - (\beta u^2 + \eta)^2 - 2u^2\beta(\beta u^2 + \eta)}{Q}$$

and simplifying

$$\beta_\omega u^4 + (\eta_\omega - I^2\beta_\omega + II_\omega\beta)\,u^2 - I^2\eta_\omega + II_\omega\eta = \frac{-3\beta^2u^4 - 4\beta\eta u^2 + 1 - \eta^2}{Q}. \tag{A.11}$$

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